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On Optimal Paths & Minimal Action

by Tasneem Zehra Husain

It sounds a bit ridiculous when you admit your jealousy of inanimate objects. If you confess that you covet the skill with which these lifeless forms navigate their circumstances, you're bound to get some strange looks. So, you keep it to yourself - for the most part. But honestly, there are times when - if you know about the least action principle - it takes all your strength to keep from declaring that you would trade places with a subatomic particle, or a ray of light, or a rubber ball, in a heartbeat. Chances are, if you know about the principle of least action, you know enough science to realize that electrons and photons and rubber balls are not active decision makers, but that doesn't keep you from envying their ability to always follow the optimal route from one point to another. In fact, it almost makes the whole thing worse. These objects are not sentient beings; it's not as if they'd suffer if they took a circuitous route! But somehow, they manage to get it right every time, whereas you - well, you often manage to take what seems like the most complicated possible life path from Point A to Point B.

So what exactly is this mysterious knowledge that subatomic particles seem to possess, and how does one go about acquiring it? We begin by recognizing that these particles aren't furiously calculating their every move, maximizing the effect thereof; they are merely obeying the laws of nature - familiar laws, like those transcribed by Newton. The least action principle offers an approach that enables us to calculate the motion of a classical object, without recourse to conventional mechanics. But this principle should not be thought of as just an alternative to Newton's laws; it is much more powerful and far deeper than that. The chief strength of the least action principle is its flexibility. It is applicable not just within the province of classical mechanics, but can be extended to the realms of optics, electronics, electrodynamics, the theory of relativity and - perhaps most shockingly - even quantum mechanics. In fact, (as is evident in Feynman's path integral formulation) the least action principle is the most logically smooth way to connect classical and quantum physics! Suffice it to say that many well known laws are encapsulated in the elegant statement that "a physical system evolves from a fixed beginning to a fixed end in such a manner that its action is minimized."

Having drummed up the anticipation, l should at least attempt to explain what the principle is, and give you a glimpse of how it works.

For starters, we need to understand what what is meant by that all important term: action. The action is the integral of a quantity called the Lagrangian, which can be thought of (for our present purposes, anyway) as the difference between the kinetic and potential energies of an object. Let's break that down. An integral is really just a sum. Kinetic energy is associated with motion, whereas potential energy is an ability conferred upon an object by virtue of its position. In fact, specifying a potential function is equivalent to stating the effect of a force; the strength of a force at any point in space is given by the slope of the potential there.

In order to avoid the complexities of calculus, let's divide the duration of our physical process into a finite number of discrete time intervals. The action is then the sum, of the value of the Lagrangian multiplied by the width of the time interval, for each time interval as the object evolves from an initial to a final state. In other words, it is the area under the curve in the figure.

If all this is getting too abstract, maybe an example will help.
Let's consider one of the simplest systems we can come up with: a classical (as opposed to quantum) particle moving in the absence of any forces. We know from Newtonian mechanics that such (inertial) bodies maintain "uniform motion in a straight line". The question is, how does the principle of least action replicate this result?

We start by writing down the action. The kinetic energy of a classical object is 1/2 mv^2, and since there is no force here, there is no potential to worry about. The action, then, is simply given by the sum of the kinetic energy of the particle in each discrete time interval, multiplied by the time interval. Since we know where and when the particle starts out, and where and when it ends up (these being the conditions that define its initial and final states), we can divide the distance d between these two points by the time taken t, to obtain the average velocity v = d/t. If the particle moved at a constant speed throughout its journey (as Newton's law says it should) this would have to be the speed it chose. The resulting action would be:

$$A_0 = \frac{1}{2}mv^2t$$

But the principle of least action says that the particle would maintain the velocity v throughout its journey only if, by doing so, the particle minimizes its action. Does the adherence to average velocity indeed guarantee minimal action?

Assume it doesn't. Assume that the action is minimized when the particle moves at a non-uniform speed. We already know the average velocity, so if the particle changes its speed, it must at some stage move faster than v, and compensate at other times by moving slower. Consider the simplest possible case (all other cases can be analyzed similarly): the particle moves faster for half the journey - say, it travels at speed (v + a) for a time t/2 - and then slows down to speed (v - a) for the remainder of the time.

The action can then be computed as below:

$$A = \left[\frac{1}{2}m(v+a)^2\right]\frac{t}{2} + \left[\frac{1}{2}m(v-a)^2\right]\frac{t}{2} = \frac{1}{2}mv^2t + \frac{1}{2}ma^2t$$

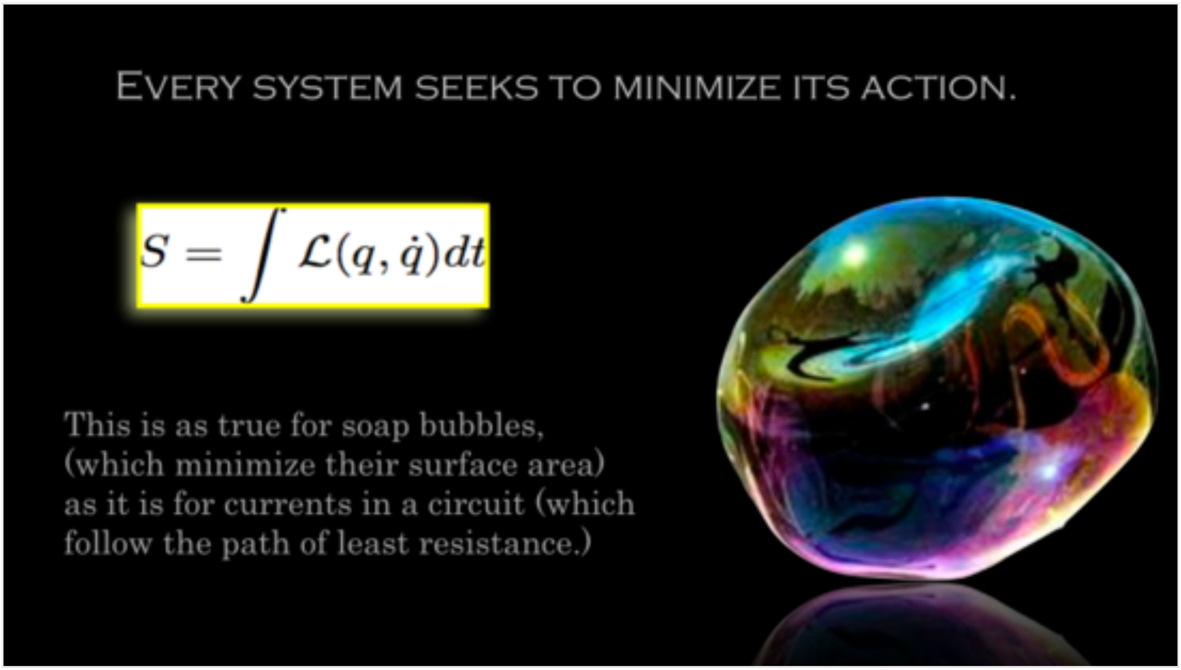
Since the square of a number is always positive, it follows that no matter how small a is, the action above will always be larger than the action that would result had the particle maintained the speed v, throughout time t.

It might seem like we are supplementing the law with additional information - a knowledge of the initial and final states. But if you think about it, we do the same in classical mechanics also; there too, we need to input two distinct pieces of data to get a sensible result from Newton's equations. Here's how:

Newton's law F = ma says that the magnitude of the force applied on a body has the same numerical value as the product of its mass, and the acceleration it experiences. In other words, given the mass of an object, and the force acting upon it, we can calculate the acceleration. But acceleration just determines the rate at which the velocity changes - it is blind to the actual values of the initial and final velocities.

When we are told that an object is accelerating at a rate of 10 m/s^2, we can conclude that with every passing second, its velocity changes by 10 m/s, but are unable to make any claims about the numerical value of the final velocity, unless we know how fast the object was moving when the force was first applied. This additional piece of information is known as an initial condition. In fact, if we want to trace the path travelled by an object under the influence of a particular force, we need two pieces of data - in addition to the initial velocity, we must also know the initial position. (The argument is similar to what we have seen above; velocity tells us how fast something moves, and in which direction - it carries no knowledge of the starting position. This information must be put in "by hand.") So, Newton's laws can be used to determine the unique path travelled by an object, as long as we know where it began its journey from, and how fast it was moving at the time.

And so, it happens that the least action principle leads to the same conclusion as Newton's familiar laws of motion. In similar vein, by writing down the appropriate Lagrangians, we can explain a host of phenomena in widely varying physical systems. The refraction of a light ray, when it passes from a rare to a dense medium, can be attributed to the fact that light "wants" to minimize the time it takes in traveling from one point to another. Since it travels more slowly in a denser medium, light will traverse a path that requires it to cross the smallest possible distance here. Physics abounds with such examples; geodesics in general relativity are merely the shortest possible paths objects can travel in a curved space-time; soap bubbles acquire shapes that minimize their surface area; currents in circuits travel the path of least resistance, and so on. The reach of the least action principle is hard to overstate.



This principle is more elegant than - for instance - Newton's laws, but it stands apart in another way also. Newton's laws, and in fact many others, are formulated in terms of differential equations; equations that are concerned with incremental changes. The path of an object is charted out by moving from point to point. At each step, you are concerned only with the next one. Instead of concerning itself with an infinitude of minutiae, the least action principle tackles the overarching problem by considering the path as a whole. It is a difference of attitude, or at the very least, perspective.

Coming back now, to us. We know where we started: in a grudging state of admiration for the unfailing instincts that guide inanimate objects along the optimal paths. Can we end up in a state where we have learnt, somehow, to do the same? It would appear not. But at least now we can make sense of the reasons why. For starters, we don't know how to write down the proper Lagrangian. Kinetic energy is puzzling enough, but the invisible potentials in which we find ourselves are often completely unknown, so we don't have an expression for the action. We don't know what it is we need to minimize.

There is yet another problem: the least action principle connects a fixed beginning to a fixed end. It only works when you know the end and you need to figure out the path taken to get there. In life, we don't really know where we will end up, leave alone when. Optimizing our trajectories might have been possible if we could step outside the bounds of space and time, and see our lives laid out as a whole. Perhaps then, we could stop obsessing about each detail along the path and simply mold the curve into a pleasing overall shape - letting the points fall where they may. But all we have is the here and now, so an incremental approach is the best we can do. And so, we inch forward step by step, focusing on the immediate, trying to make the most of the moment.

Maybe it is just as well. Perhaps for us sentient beings, the goal is not simply to get from Point A to Point B. Our haphazard Brownian motion through life, that causes us to scatter off unexpected obstacles and collide with unforeseen objects, also makes our hearts expand and forces our minds to grow. Perhaps that is the point. Maybe for us, it really is about the journey, and not the destination.

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